ON THE PLANE DEFORMATION OF A RIGID-PLASTIC BODY

(K TEORII PLOSKOI DEFORMATSII ZHESTKO-PLASTICHESKOGO TELA)

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Plane plastic flow of a rigid-plastic body is analyzed. As a coordinate system the flow lines and the curves orthogonal to them are selected. The analogues of the Hencky integrals taken along these lines are presented. The compatibility equation of the stress and the velocity fields is derived, and a method of obtaining various solutions corresponding to the assumed flow fields which follows from this equation is indicated. The relationship between the compatibility equation and the extremal properties of a true velocity field is studied, together with certain velocity classes for which the flow lines coincide with the slip lines and the trajectories of principal stresses.

1. The plane plastic flow of a rigid-plastic body is described, as is well known [1,2], by the following equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$(\sigma_x - \sigma_y)^2 + 4\tau^2_{xy} = 4k^2 \tag{1.1}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \qquad \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{\partial v_y / \partial x + \partial v_x / \partial y}{\partial v_x / \partial x - \partial v_y / \partial y}$$

Let the equations for the flow lines and the orthogonal curves be

$$q_1 = q_1(x, y) = \text{const}, \qquad q_2 = q_2(x, y) = \text{const} \tag{1.2}$$

In curvilinear orthogonal coordinates q_1, q_2 (1.1) has the following form (e.g. [3]):

$$\frac{\partial}{\partial q_1}(H_2\sigma_{11}) + \frac{\partial}{\partial q_2}(H_1\sigma_{12}) + \frac{\partial H_1}{\partial q_2}\sigma_{12} - \frac{\partial H_2}{\partial q_1}\sigma_{22} = 0$$

$$\frac{\partial}{\partial q_1}(H_2\sigma_{12}) + \frac{\partial}{\partial q_2}(H_1\sigma_{22}) + \frac{\partial H_2}{\partial q_1}\sigma_{12} - \frac{\partial H_1}{\partial q_2}\sigma_{11} = 0 \tag{1.3}$$

$$(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2 = 4k^2$$

$$\xi_{11} + \xi_{22} = 0, \qquad \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}} = \frac{\xi_{12}}{\xi_{11} - \xi_{22}}$$

where σ_{ij} are stress components in the given coordinate system; k is the limiting shear stress; ξ_{ij} are deformation velocity [strain rate] components; H_1 and H_2 are Lamé's constants.

The deformation velocity in the q_1, q_2 system satisfies

$$\xi_{11} = \frac{1}{H_1} \frac{\partial v}{\partial q_1}, \qquad \xi_{22} = \frac{1}{H_1 H_2} \frac{\partial H_2}{\partial q_1} v, \ \xi_{12} = \frac{H_1}{H_2} \frac{\partial}{\partial q_2} \frac{v}{H_1} \tag{1.4}$$

where v is the modulus of the velocity vector. Introduce now new variables

$$\sigma = \frac{1}{2}(\sigma_{11} + \sigma_{22}), \qquad \left.\begin{matrix}\sigma_{11}\\ \sigma_{22}\end{matrix}\right\} = \sigma \pm k \cos 2\beta, \quad \sigma_{12} = k \sin 2\beta \tag{1.5}$$

where β is an angle formed by the velocity vector and the direction of the larger principal stress. Thus (1.3), taking into account (1.4) and (1.5), can be written as

$$\frac{\partial \sigma}{\partial q_1} + k \frac{\partial \cos 2\beta}{\partial q_1} + \frac{2k \sin 2\beta}{H_2} \frac{\partial H_1}{\partial q_2} + \frac{kH_1}{H_2} \frac{\partial \sin 2\beta}{\partial q_2} + \frac{2k \cos 2\beta}{H_2} \frac{\partial H_2}{\partial q_1} = 0 \tag{1.6}$$

$$\frac{\partial \sigma}{\partial q_2} + \frac{kH_2}{H_1} \frac{\partial \sin 2\beta}{\partial q_1} - \frac{2k \cos 2\beta}{H_1} \frac{\partial H_1}{\partial q_2} - k \frac{\partial \cos 2\beta}{\partial q_2} + \frac{2k \sin 2\beta}{H_1} \frac{\partial H_2}{\partial q_1} = 0 \tag{1.7}$$

$$v = \frac{f(q_2)}{H_2} \tag{1.8}$$

$$\tan 2\beta = \frac{H_1}{2\partial H_2 / \partial q_1} \frac{\partial}{\partial q_2} \ln \frac{H_1 H_2}{f(q_2)} = F \tag{1.9}$$

Here $f(q_2)$ is some function of its argument. The Lamé equation

$$\frac{\partial}{\partial q_1}\left(\frac{1}{H_1} \frac{\partial H_2}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{1}{H_2} \frac{\partial H_1}{\partial q_2}\right) = 0 \tag{1.10}$$

should supplement Equations (1.6) to (1.9).

2. We shall derive now the analogues of Hencky's integrals for the equations along the flow lines and the curves orthogonal to them

$$\sigma + k \cos 2\beta = -\int\left(\frac{2kF_1}{H_2} \frac{\partial H_1}{\partial q_2} + \frac{kH_1}{H_2} \frac{\partial F_1}{\partial q_2} + \frac{2kF_2}{H_2} \frac{\partial H_2}{\partial q_1}\right) dq_1 + \eta(q_2) \tag{2.1}$$

$$\sigma - k \cos 2\beta = \int\left(\frac{2kF_2}{H_1} \frac{\partial H_1}{\partial q_2} - \frac{kH_2}{H_1} \frac{\partial F_1}{\partial q_1} - \frac{2kF_1}{H_1} \frac{\partial H_2}{\partial q_1}\right) dq_2 + \gamma(q_1) \tag{2.2}$$

Here $F_1 = \sin 2\beta$, $F_2 = \cos 2\beta$ and are determined by (1.9); $\eta(q_2)$, $\gamma(q_1)$ are some functions.

We shall assume in the sequel that the derivatives of the functions in the above equations exist and are continuous. Eliminating by differentiating the function σ from (1.6) and (1.7), we obtain

$$2\frac{\partial^2 \cos 2\beta}{\partial q_1 \partial q_2} + \frac{\partial \sin 2\beta}{\partial q_2}\left(\frac{2}{H_2}\frac{\partial H_1}{\partial q_2} + \frac{\partial}{\partial q_2}\frac{H_1}{H_2}\right) - \frac{\partial \sin 2\beta}{\partial q_1}\left(\frac{\partial}{\partial q_1}\frac{H_2}{H_1} + \frac{2}{H_1}\frac{\partial H_2}{\partial q_1}\right) +$$
$$+ 2\frac{\partial \cos 2\beta}{\partial q_2}\frac{1}{H_2}\frac{\partial H_2}{\partial q_1} + \frac{2}{H_1}\frac{\partial H_1}{\partial q_2}\frac{\partial \cos 2\beta}{\partial q_1} + \frac{H_1}{H_2}\frac{\partial^2 \sin 2\beta}{\partial q_2^2} -$$
$$- \frac{H_2}{H_1}\frac{\partial^2 \sin 2\beta}{\partial q_1^2} + 4\sin 2\beta \frac{\partial}{\partial q_2}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial q_2}\right) +$$
$$+ 2\cos 2\beta\left[\frac{\partial}{\partial q_2}\left(\frac{1}{H_2}\frac{\partial H_2}{\partial q_1}\right) + \frac{\partial}{\partial q_1}\left(\frac{1}{H_1}\frac{\partial H_1}{\partial q_2}\right)\right] = 0 \qquad (2.3)$$

or, considering (1.9)

$$(4F^2 - 2)\frac{\partial F}{\partial q_1}\frac{\partial F}{\partial q_2} + (1 + F^2)\left[\frac{\partial F}{\partial q_2}\left(\frac{2}{H_2}\frac{\partial H_1}{\partial q_2} + \frac{\partial}{\partial q_2}\frac{H_1}{H_2}\right) - 2F\frac{\partial^2 F}{\partial q_1 \partial q_2} -\right.$$
$$- \frac{\partial F}{\partial q_1}\left(\frac{\partial}{\partial q_1}\frac{H_2}{H_1} + \frac{2}{H_1}\frac{\partial H_2}{\partial q_1}\right) - \frac{2F}{H_2}\frac{\partial F}{\partial q_2}\frac{\partial H_2}{\partial q_1} - \frac{2F}{H_1}\frac{\partial F}{\partial q_1}\frac{\partial H_1}{\partial q_2} +$$
$$\left. + \frac{H_1}{H_2}\frac{\partial^2 F}{\partial q_2^2} - \frac{H_2}{H_1}\frac{\partial^2 F}{\partial q_1^2}\right] - 3F\left[\left(\frac{\partial F}{\partial q_2}\right)^2\frac{H_1}{H_2} - \left(\frac{\partial F}{\partial q_1}\right)^2\frac{H_2}{H_1}\right] +$$
$$+ (1 + F^2)^2\left[4F\frac{\partial}{\partial q_2}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial q_2}\right) + \frac{\partial}{\partial q_2}\left(\frac{2}{H_2}\frac{\partial H_2}{\partial q_1}\right) + \frac{\partial}{\partial q_1}\left(\frac{2}{H_1}\frac{\partial H_1}{\partial q_2}\right)\right] = 0 \quad (2.4)$$

Function F in (2.3) and (2.4) is determined from (1.9); moreover

$$\frac{\partial \sin 2\beta}{\partial q_i} = \frac{1}{(1 + F^2)^{3/2}}\frac{\partial F}{\partial q_i}, \qquad \frac{\partial \cos 2\beta}{\partial q_i} = -\frac{F}{(1 + F^2)^{3/2}}\frac{\partial F}{\partial q_i}$$
$$\frac{\partial^2 \sin 2\beta}{\partial q_i^2} = \frac{1}{(1 + F^2)^{5/2}}\left[\frac{\partial^2 F}{\partial q_i^2}(1 + F^2) - 3F\left(\frac{\partial F}{\partial q_i}\right)^2\right] \qquad \left(\begin{matrix} i = 1, 2 \\ j = 1, 2 \end{matrix} \quad i \neq j\right)$$
$$\frac{\partial^2 \cos 2\beta}{\partial q_i \partial q_j} = \frac{1}{(1 + F^2)^{5/2}}\left[(2F^2 - 1)\frac{\partial F}{\partial q_i}\frac{\partial F}{\partial q_j} - F(1 + F^2)\frac{\partial^2 F}{\partial q_i \partial q_j}\right] \qquad (2.5)$$

Equation (2.4) is the third-order equation relative to H_1, H_2 and $f(q_2)$, and it represents the compatibility equation for the stress and velocity fields. This can be formulated in the following theorem:

Theorem. A necessary and sufficient condition for a flow-line field, which is determined by the Lamé constants, having continuous derivatives up to third order, to be a true flow-line field is that there exists such a function $f(q_2)$ which after the substitution of H_1 and H_2 into

(2.4) satisfies the compatibility equation identically.

Note that for $\partial H_2/\partial q_1 = 0$ the compatibility equation in the form (2.3) reduces, in view of (1.9) and (1.10), to an identity. In the case, however, when we have simultaneously

$$\frac{\partial H_2}{\partial q_1} = 0, \qquad \frac{\partial}{\partial q_2} \ln \frac{H_1 H_2}{f(q_2)} = 0$$

the components of the velocity vector are identically zero.

For the true flow-line field (2.4) is generally an equation of the third order with respect to $f(q_2)$. The order of this equation may be reduced to the second. This fact permits the solution corresponding to the given flow-line field to be found in the following way. The function F is determined from (1.9), it is then substituted into (2.4), from which, in turn, we find $f(q_2)$, provided that the conditions of the theorem are satisfied. Next, $\tan 2\beta$ is found from (1.8) and (1.9). Finally σ is found from (2.1) and (2.2).

Example. Consider curvilinear orthogonal coordinates with the Lamé coefficients

$$H_1 = c_1 \exp(aq_1 + bq_2), \qquad H_2 = c_2 \exp(aq_1 + bq_2) \tag{2.6}$$

where a, b and c_i are constants. (2.6) represents two families of logarithmic spirals. From (1.9) we obtain

$$\tan 2\beta = F = \frac{c_1}{2c_2 a}\left[2b - \frac{d \ln f(q_2)}{dq_2}\right] \tag{2.7}$$

From (2.6) and (2.7) it follows that the compatibility equation is reduced to an ordinary differential equation for $f(g_2)$. Thus the conditions of the above theorem are fulfilled and the compatibility equation according to (2.3), (2.6) and (2.7) is

$$\frac{2c_1 b}{c_2} \frac{d \sin 2\beta}{dq_2} + \frac{2a \, d\cos 2\beta}{dq_2} + \frac{c_1}{c_2} \frac{d^2 \sin 2\beta}{dq_2{}^2} = 0 \tag{2.8}$$

or

$$bm \sin 2\beta + a \cos 2\beta + \frac{m}{2} \frac{d \sin 2\beta}{dq_2} = c \tag{2.9}$$

hence

$$q_2 = m \int \frac{\cos 2\beta \, d\beta}{c - bm \sin 2\beta - a \cos 2\beta} \qquad \left(m = \frac{c_1}{c_2}\right)$$

$$q_2 = -\frac{ma\beta}{a^2 + b^2 m^2} - \frac{bm^2}{2(a^2 + b^2 m^2)} \ln(c - a\cos 2\beta - bm \sin 2\beta) + \frac{acm}{a^2 + b^2 m^2} p + D_1 \tag{2.10}$$

where c and D_i are constants:

$$p = \begin{cases} \dfrac{1}{\sqrt{c^2 - a^2 - b^2m^2}} \tan^{-1} \dfrac{(c + a)\tan\beta - bm}{\sqrt{c^2 - a^2 - b^2m^2}} & (c^2 - a^2 - b^2m^2 > 0) \\ -\dfrac{1}{\sqrt{a^2 + b^2m^2 - c^2}} \operatorname{Arth} \dfrac{(c + a)\tan\beta - bm}{\sqrt{a^2 + b^2m^2 - c^2}} & (a^2 + b^2m^2 - c^2 > 0) \end{cases} \quad (2.11)$$

Because of (1.6) and (2.6) the integral (2.1) taken along the flow line is

$$\sigma + 2kcq_1 = \eta(q_2) \quad (2.12)$$

Let us now calculate the integral (2.2) taken along the lines orthogonal to the flow lines. In doing so we rewrite (2.2), taking into account (1.7) and (2.9), in the following form:

$$\sigma - \frac{2kbc}{a} q_2 - k\cos 2\beta + \frac{kmb\sin 2\beta}{a} + \frac{2k(a^2 + b^2m^2)}{am} \int \sin 2\beta \, dq_2 = \gamma(q_1)$$

It follows from (2.10) that

$$\frac{2k(a^2 + b^2m^2)}{am} \int \sin 2\beta \, dq_2 = \frac{2k(a^2 + b^2m^2)}{a} \int \frac{\cos 2\beta \sin 2\beta \, d\beta}{c - bm\sin 2\beta - a\cos 2\beta}$$

Hence

$$\frac{2k(a^2 + b^2m^2)}{am} \int \sin 2\beta \, dq_2 = -\frac{4kbcm}{a^2 + b^2m^2}\beta + \frac{k(a^2c - b^2m^2c)}{a(a^2 + b^2m^2)} \ln(c - bm\sin 2\beta - a\cos 2\beta) + \frac{k}{a}(a\cos 2\beta - bm\sin 2\beta) + \frac{2km}{a^2 + b^2m^2}(2bc^2 - a^2b - b^3m^2)p + D_2$$

Thus along the line q_1 = const the following is satisfied:

$$\sigma - \frac{2kbc}{a} q_2 - \frac{4kbcm}{a^2 + b^2m^2}\beta + \frac{k(a^2c - b^2cm^2)}{a(a^2 + b^2m^2)} \ln(c - bm\sin 2\beta - a\cos 2\beta) + \frac{2km}{a^2 + b^2m^2}(2bc^2 - b^3m^2 - a^2b)p = \gamma(q_1) \quad (2.13)$$

From (2.12) and (2.13) it follows that

$$\sigma = -2kcq_1 + \frac{2kbc}{a} q_2 + \frac{4kbmc}{a^2 + b^2m^2}\beta - \frac{k(a^2c - b^2cm^2)}{a(a^2 + b^2m^2)} \ln(c - bm\sin 2\beta - a\cos 2\beta) - \frac{2km}{a^2 + b^2m^2}(2bc^2 - b^3m^2 - a^2b)p + D_2 \quad (2.14)$$

where p is determined from (2.11).

From (1.9) it follows that

$$\ln f(q_2) = 2bq_2 - \frac{2a}{m}\int \tan 2\beta \, dq_2 + D_3$$

since

$$\int \tan 2\beta \, dq_2 = m \int \frac{\sin 2\beta \, d\beta}{c - bm \sin 2\beta - a \cos 2\beta} = \frac{am}{2(a^2 + b^2m^2)} \ln (c - bm \sin 2\beta - a \cos 2\beta) - $$
$$- \frac{bm^2}{a^2 + b^2m^2}\beta + \frac{bcm^2}{a^2 + b^2m^2} p + D_3$$

therefore

$$f(q_2) = \exp\left[2bq_2 - \frac{a^2}{a^2 + b^2m^2} \ln (c - bm \sin 2\beta - a \cos 2\beta) + \right.$$
$$\left. + \frac{2abm}{a^2 + b^2m^2}\beta - \frac{2abcm}{a^2 + b^2m^2} p + D_3\right] \tag{2.15}$$

and thus, according to (1.8)

$$v = \exp\left[bq_2 - \frac{a^2}{a^2 + b^2m^2} \ln (c - bm \sin 2\beta - a \cos 2\beta) + \right.$$
$$\left. + \frac{2abm}{a^2 + b^2m^2}\beta - aq_1 - \frac{2abcm}{a^2 + b^2m^2} p + D_3\right] \tag{2.16}$$

The relationships (2.10), (2.14) and (2.16) determine plastic flow which corresponds to the flow lines in the form of logarithmic spirals (2.16). Such a flow can be visualized in an extrusion of a plastic medium through a channel, the walls of which are logarithmic spirals, and the tangential stresses along these walls are constant.

If in the above relations we put $a = m = 1$, $b = 0$, then Nádái's solution for the radial flow lines is obtained.

It is easy to verify that the conditions of the theorem are satisfied by the following class of curvilinear coordinates:

$$H_1 = \Phi'(q_1)\Psi(q_2), \qquad H_2 = \Phi(q_1)\Psi'(q_2) \tag{2.17}$$

where $\Phi(q_1)$ and $\Psi(q_2)$ are arbitrary functions having continuous derivatives up to the third order. The non-admissible coordinates, for example, are

$$H_1 = H_2 = H = c \exp(2mq_1q_2) \qquad (c, m = \text{const}) \tag{2.18}$$

3. Let some flow-line field satisfy the conditions of the theorem. Consider some plastic region ω and given flow-line field in it. We shall assume that along the contour of ω the velocities are given by (1.8):

$$v = \frac{f(q_2)}{H_2}$$

In this case, in view of the well-known theorem of extremal properties of a true velocity field [1], the function $f(q_2)$ must yield the minimum of the functional

$$I = \sqrt{2}k \int_\omega \sqrt{2\xi_{11}^2 + \frac{1}{2}\xi_{12}^2}\, d\omega \tag{3.1}$$

which, after integrating along q_1 and in view of (1.4) and (1.8), has the form

$$I = \sqrt{2}k \int_{q_2=a}^{q_2=b} Q\,[q_2, f(q_2), f'(q_2)]\, dq_2 \tag{3.2}$$

with the usual transverse conditions

$$[Q_{f'}]_{q_2=a} = [Q_{f'}]_{q_2=b} = 0 \tag{3.3}$$

where $Q(f, f', q_2)$ is known. Thus for the determination of $f(q_2)$ a direct method can be applied.

4. Consider the case when $\partial H_2/\partial q_1 = 0$. In view of the known relationships we have

$$\frac{\partial\alpha}{\partial s_1} = -\frac{1}{H_2 H_1}\frac{\partial H_1}{\partial q_2}, \qquad \frac{\partial\alpha}{\partial s_2} = \frac{1}{H_1 H_2}\frac{\partial H_2}{\partial q_1}, \qquad \frac{\partial\alpha}{\partial s_2} = 0. \tag{4.1}$$

where a is an angle formed by the tangent of the flow line with a variable direction; $\partial a/\partial s_i$ is the curvature of the coordinate lines. Consequently, according to (4.1), the flow lines are equidistant curves. On the other hand, it follows from (1.9) that $\beta = 0 \pm \pi/4$, i.e. the flow lines coincide with the slip lines. Clearly, the converse is also true. Thus the necessary and sufficient condition for the coincidence of the flow lines with slip lines is that the flow lines be equidistant curves; clearly the motion of a medium as a rigid body is excluded.

5. Consider now the case when the flow lines coincide with the directions of the principal stresses. Since in this case $\beta = 0 \pm \pi/2$, and because of (1.9), we have

$$\frac{\partial}{\partial q_2}\ln\frac{H_1 H_2}{f(q_2)} = 0 \tag{5.1}$$

On the other hand (1.6) and (1.7) have the following well-known form:

$$\frac{\partial\sigma}{\partial q_1} + \frac{2k}{H_2}\frac{\partial H_2}{\partial q_1} = 0, \qquad \frac{\partial\sigma}{\partial q_2} - \frac{2k}{H_1}\frac{\partial H_1}{\partial q_2} = 0 \tag{5.2}$$

or

$$\sigma + 2k\ln H_2 = \eta(q_2), \qquad \sigma - 2k\ln H_1 + \gamma(q_1) \tag{5.3}$$

Hence in view of (5.1)

$$\frac{1}{4k}\eta(q_2) = \ln f(q_2) + c \qquad (c = \text{const}) \tag{5.4}$$

Thus if the flow lines coincide with the trajectories of principal stresses, then the following is true:

$$\ln H_1H_2 = \ln f(q_2) - \frac{1}{4k}\gamma(q_1) + c \tag{5.5}$$

Obviously, the converse is also true, i.e. (5.5) represents a necessary and sufficient condition for the coincidence of the flow lines and the trajectories of the principal stresses.

As an example consider an isometric net of the flow lines (or the trajectories of the principal stresses). It follows from (1.10) and (5.5) in this case that

$$[\ln f(q_2)]'' = \frac{1}{4k}\gamma''(q_1) = n \qquad (n = \text{const}) \tag{5.6}$$

Therefore

$$H_1 = H_2 = H = \exp\left[\frac{1}{2}\left(\frac{nq_2^2}{2} + c_1q_2 - \frac{nq_1^2}{2} - c_3q_1 + c_2\right)\right]$$
$$\sigma = k\,(nq_2^2 + 2c_1q_2 + nq_1^2 + 2c_3q_1 + c_4) \qquad (c_i = \text{const}) \tag{5.7}$$
$$v = \exp\left(\frac{n}{4}q_2^2 + \frac{n}{4}q_1^2 + \frac{c_1}{2}q_2 + \frac{c_3}{2}q_1 + c\right)$$

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